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Degeneracy in Mean Nonlinear Approximation

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Let F be an approximating function with parameters taken from a parameter space P such that $F(A, \cdot) \in C[a, b]$ for all $A \in P$. Let $\int$ denote integration on $[a, b]$, and for $g \in C[a, b]$ define

$$\| g \| = \int | g |.$$

The approximation problem is: given $f \in C[a, b]$ to find a parameter $A^* \in P$ for which $\| f - F(A^*, \cdot)\|$ is minimal. Any such parameter A^* is called best.

The case where F is an (ordinary) rational approximating function is covered in [1]; this paper is an extension of the latter.

1. PRELIMINARIES

A fundamental role in mean approximation is played by the set

$$Z(A) = \{x : f(x) = F(A, x), a \leqslant x \leqslant b\}.$$

Let $\sim Z(A)$ denote $[a, b] \sim Z(A)$. We will make use of the characterization lemma for linear mean approximation, a proof of which appears in [3, p.103].

LEMMA 1. *A necessary and sufficient condition that*

$$\| f - F(A, \cdot)\| \leqslant \| f - F(A, \ ) - \lambda h \| \tag{1}$$

for all λ is that

$$\left| \int h \operatorname{sgn}(f - F(A, \cdot)) \right| \leqslant \int_{Z(A)} | h |. \tag{2}$$

If strict inequality occurs in (2), then strict inequality occurs in (1) for all nonzero λ.

2. Zero a Best Approximation

In general, because of the nonconvexity of $\{F(A, \cdot) : A \in P\}$, there is no simple test for an approximation being best. In the special case where $\{F(A, \cdot) : A \in P\}$ is closed under scalar multiplication and the approximation is zero, the above lemma completely answers this question, namely 0 is best to f if and only if for all $A \in P$,

$$\operatorname{sgn} \int_{\sim Z(0)} | F(A, \cdot)| < \int_{Z(0)} | F(A, \cdot)|$$

and 0 is a unique best approximation if the inequality is strict for all A not corresponding to zero.

Example. Let $\{F(A, \cdot): A \in P\}$ be closed under scalar multiplication and $| F(A, \cdot)|$ be convex for all $A \in P$. Select f such that $Z(0)$ contains

$$[a, a + (b - a)/3] \cup [a + 2(b - a)/3, b].$$

By convexity of $| F(A, \cdot)|$ we have

$$\int_{\sim Z(0)} | F(A, .) | < \int_{Z(0)} | F(A, .) |$$

for all A not corresponding to zero, hence 0 is a unique best approximation.

Approximating functions of the form $F(A, x) = a_1\phi(a_2 x)$, ϕ convex, $\phi \geqslant 0$ satisfy the hypotheses of the example. A special case is where $F(A, x) = a_1 \exp(a_2 x)$. We can replace convexity of $| F(A, \cdot)|$ in the example by monotonicity of $| F(A, \cdot)|$.

3. Degeneracy

Definition. The *sum space* of $F(A, \cdot)$ is the set of functions h such that $F(A, \cdot) + \lambda h \in \{F(B, \cdot): B \in P\}$ for all $| \lambda |$ sufficiently small.

Consider the case in which approximants are of the form

$$F(A, x) = \sum_{k=1}^{n} a_k\phi_k(x) + \sum_{k=1}^{m} a_{n+k}\psi_k(a_{n+m+k}x), \qquad \alpha_k < a_{n+m+k} < \beta_k\,, \quad (3)$$

where ψ_k is a nonconstant function. We say that an approximant is *degenerate* in ψ_k if it can be written in the form (3) with $a_{n+k} = 0$. Its sum space then includes $\{\psi_k(\gamma x): \alpha_k < \gamma < \beta_k\}$.

4. Accumulation Points of $Z(A)$

THEOREM 1. *Let c be a point such that for any interval I containing c, an element h of the sum space of F(A, ·) exists such that $h(x) \geqslant 0$ for $x \in I$ and*

$$\int_I h > \int_W |h| \qquad W = [a, b] \sim I \tag{4}$$

Let f have F(A, ·) as best approximation. Then c is an accumulation point of Z(A).

Proof. Suppose c is not an accumulation point of $Z(A)$. Then there exists a nondegenerate interval I with c as one endpoint such that $f - F(A, \cdot)$ does not vanish in the interior of I. Assume without loss of generality that $I = [c, \mu]$ and $f - F(A, \cdot)$ is positive on (c, μ). Select h in the sum space of $F(A, \cdot)$ satisfying (4). We have

$$\left| \int h \cdot \operatorname{sgn}(f - F(A, \cdot)) \right| \geqslant \int_c^\mu h - \int_W |h| \cdot |\operatorname{sgn}(f - F(A, \cdot))|$$

$$\left| \int h \cdot \operatorname{sgn}(f - F(A, \cdot)) \right| - \int_{Z(A)} |h| \geqslant \int_c^\mu h - \int_W |h| > 0.$$

It follows by Lemma 1 that $F(A, \cdot)$ is not best, proving Theorem 1.

COROLLARY. *Let there exist for given η, μ, $a < \eta < \mu < b$, an element $h \geqslant 0$ of the sum space of F(A, ·) such that*

$$\int_\eta^\mu h > \int_W h, \qquad W = [a, b] \sim [\eta, \mu].$$

F(A, ·) is a best approximation only to itself.

In the case $F(A, \cdot)$ is a polynomial rational function of degeneracy 2, such an element h exists [1, Theorem 1].

5. Positive Measure

THEOREM 2. *Let the orthogonal complement of the sum space of F(A, ·) in the space of bounded measurable functions be the functions vanishing almost everywhere. Then F(A, ·) is best to f only if Z(A) has positive measure.*

Proof. Suppose $F(A, \cdot)$ is best to f and $Z(A)$ is a set of measure zero. By Lemma 1, for all h in the sum space of $F(A, \cdot)$,

$$\left| \int h \operatorname{sgn}(f - F(A, \cdot)) \right| \leqslant \int_{(\ )} |h| = 0.$$

This is true for all h in the sum space of $F(A, \cdot)$ and so $\operatorname{sgn}(f - F(A, \cdot))$ is in the orthogonal complement of the sum space. As $\operatorname{sgn}(f - F(A, \cdot))$ is bounded and measurable, it follows that $\operatorname{sgn}(f - F(A, \cdot)) = 0$ almost everywhere, hence $f \equiv F(A, \cdot)$. We have a contradiction and the theorem is proven.

In [2] are given conditions for sum spaces containing $\{\phi(\alpha x): -\mu < \alpha < \mu\}$ to have functions vanishing almost everywhere as the orthogonal complement.

6. Approximation on a Finite Subset

We briefly consider approximation on a finite subset $\{x_1, \ldots, x_n\}$, $x_1 < \cdots < x_n$, with norm

$$\| g \| = \sum_{k=1}^{n} | g(x_k)| \, w_k ,$$

$w_1, \ldots, w_k$ being positive weights. Define

$$Z(A) = \{x : f(x) = F(A, x), x = x_1, \ldots, x_n\}.$$

The analog of Lemma 1 for equal weights is given by Rice [3, p. 114]. By using arguments similar to those of Section 2 we obtain the following.

Example. Let $\{F(A, \cdot): A \in P\}$ be closed under scalar multiplication and $| F(A, \cdot)|$ be strictly monotonic or identically zero for all $A \in P$. Let all weights be equal and $n \geqslant 3$. Select f such that $Z(0)$ contains all but one point of $\{x_1, \ldots, x_n\}$ and that point is not x_1 or x_n. 0 is a unique best approximation to f.

The analog of Theorem 1 is the following.

Theorem 3. *Suppose for an index j of $1, \ldots, n$ there is an element h of the sum space of $F(A, \cdot)$ such that $h(x_j) > 0$ and*

$$h(x_j) \, w_j > \sum_{\substack{k=1 \\ k \neq j}}^{n} | h(x_k)| \, w_k$$

If $F(A, \cdot)$ is best to f then $x_j \in Z(A)$.

Corollary. *If the above theorem holds for all indices, $F(A, \cdot)$ best to f implies $f \equiv F(A, \cdot)$.*

The analog of Theorem 2 is the following.

Theorem 4. *Let the orthogonal complement of the sum space of $F(A, \cdot)$*

in the space of functions on $\{x_1, ..., x_n\}$ *be zero. Then* $F(A, \cdot)$ *is best to* f *only if* $Z(A)$ *is nonempty.*

7. Exponential Approximation

Let V_n be the family of functions of the form

$$F(A, x) = \sum_{k=1}^{n} a_k \exp(a_{n+k}x), \qquad a_{n+i} \neq a_{n+j} \qquad \text{if} \quad i \neq j.$$

If m of the coefficients $a_1, ..., a_n$ vanish, then $F(A, \cdot)$ is said to have degeneracy m. $F(A, \cdot)$ is *degenerate* if it has positive degeneracy. The sum space of a degenerate element contains $\{\exp(\alpha x)\colon \alpha \text{ real}\}$.

The zero function is the only degenerate element of V_1 and has degeneracy 1 with respect to that family. By the discussion of Section 2, there exist non-zero continuous f such that 0 is a unique best approximation in V_1. An open question is whether for $n \geqslant 2$ there exists $f \notin V_n$ with a best approximation of degeneracy 2 or more.

Let $\mu \in (a, b)$ be given and let

$$h_k(x) = \exp(k\mu) \cdot \exp(-kx) = \exp(k(\mu - x)),$$

then

$$h_k(x) \to \infty, \qquad x < \mu,$$
$$h_k(x) \to 0, \qquad x > \mu,$$

hence $\int_a^\mu h_k > \mu - a$ and $\int_\mu^b h_k \to 0$. Thus the hypotheses of Theorem 1 are satisfied for k sufficiently large. A similar hypothesis is satisfied for the endpoint b, so we have the following.

Theorem 5. *Let* $F(A, \cdot)$ *be a degenerate element of* V_n. *Then* a *and* b *are accumulation points of* $Z(A)$.

It is shown in [2] that the orthogonal complement of $\{\exp(\alpha x)\colon -\mu < \alpha < \mu\}$ is zero, hence by Theorem 2, we have the following.

Theorem 6. *Let* $F(A, \cdot)$ *be a degenerate element of* V_n, *then* $Z(A)$ *is of positive measure.*

An immediate consequence of Theorem 5 or 6 is the following.

Corollary. *If* $F(A, \cdot)$ *is a degenerate element of* V_n, *the only analytic function which has* $F(A, \cdot)$ *as best approximation on* $[a, b]$ *is* $F(A, \cdot)$ *itself.*

Let us now consider approximation on finite subsets. From Theorem 3 we obtain the following.

THEOREM 7. *Let $F(A, \cdot)$ be a degenerate best approximation to f by V_n on $\{x_1, ..., x_m\}$, then x_1 and x_m are in $Z(A)$.*

As before, the question of whether there exists $f \notin V_n$ with a best approximation of degeneracy 2 or more is open.

8. *Rational Approximation on a Finite Point Set*

Let us first consider approximation by $R_m^n[a, b]$ on $\{x_1, ..., x_N\} \subset [a, b]$. As all nonzero elements of $R_1^0[a, b]$ are strictly monotonic and of one sign, it follows from a result in section 6 that for $N \geqslant 3$, there is $f \neq 0$ with 0 as unique best approximation in $R_1^0[a, b]$. Hence there exist degenerate best approximations. Let r be an element of $R_m^n[a, b]$ of degeneracy 1. Then there exists h satisfying the hypothesis of Theorem 3 with index 1 if $x_1 = a$ and for index N if $x_N = b$, hence we have the following.

THEOREM 8. *Let r be an element of $R_m^n[a, b]$ of degeneracy 1. Let r be best to f in $R_m^n[a, b]$ on $\{a, x_2, ..., x_{N-1}, b\}$, then $Z(r)$ contains a and b.*

If r is an element of $R_m^n[a, b]$ of degeneracy 2, then there exists h satisfying the hypothesis of Theorem 3 for all indices. Hence by the corollary, we have the following.

THEOREM 9. *Let r be an element of $R_m^n[a, b]$ of degeneracy 2 or more and r be best in $R_m^n[a, b]$ to f on $\{x_1, ..., x_N\}$, then $f \equiv r$.*

Let us next consider approximation by ratios of polynomials of degree n to polynomials of degree m on $\{x_1, ..., x_N\}$. If r is a degenerate ratio, then h exists satisfying the hypothesis of Theorem 3 for all indices; hence r best to f implies $f \equiv r$.

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